ON RESTRICTION OF ROOTS ON AFFINE T-VARIETIES

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Abstract. Let X be a normal affine algebraic variety with regular action of a torus $\mathbb{T}$ and $T \subset \mathbb{T}$ be a subtorus. We prove that each root of X with respect to T can be obtained by restriction of some root of X with respect to $\mathbb{T}$. This allows to get an elementary proof of the description of roots of the affine Cremona group. Several results on restriction of roots in the case of subtorus action on an affine toric variety are obtained.

Introduction

Let $\mathbb{T}$ be an algebraic torus. An algebraic variety endowed with an effective action of $\mathbb{T}$ is called a $\mathbb{T}$-*variety*. Let X be an affine $\mathbb{T}$-variety. The algebra $A = \mathbb{K}[X]$ of regular functions on X is graded by the lattice M of characters of the torus $\mathbb{T}$. A derivation ∂ on an algebra A is said to be *locally nilpotent* (LND) if for each $a \in A$ there exists $n \in \mathbb{N}$ such that $\partial^n(a) = 0$. A locally nilpotent derivation on A is said to be *homogeneous* if it respects the M-grading. A homogeneous LND ∂ shifts the M-grading by some lattice vector $\deg \partial$. This vector is named the *degree* of ∂. The degrees of homogeneous LNDs are called $\mathbb{T}$-*roots* of the $\mathbb{T}$-variety X. This definition imitates in some sense the notion of a root from Lie Theory. Let $\mathbb{G}_a$ be the additive group of the ground field $\mathbb{K}$. It is well known that LNDs on A are in bijective correspondence with regular actions of $\mathbb{G}_a$ on X. Namely, for given LND ∂ on A the map $\varphi_\partial : \mathbb{G}_a \times A \to A$, $\varphi_\partial(t, \partial) = \exp(t\partial)(f)$, defines a $\mathbb{G}_a$-action, and any regular $\mathbb{G}_a$-action on X arises in this way. It is easy to see that an LND on A is homogeneous if and only if the corresponding $\mathbb{G}_a$-action is normalized by the torus $\mathbb{T}$ in the group $\mathrm{Aut}(X)$. In these terms, a root is a character by which $\mathbb{T}$ acts on $\mathbb{G}_a$.

Any derivation on $\mathbb{K}[X]$ extends to a derivation on the field of fractions $\mathbb{K}(X)$ by the Leibniz rule. A homogeneous LND ∂ on $\mathbb{K}[X]$ is said to be of *fiber type* if $\partial(\mathbb{K}(X)^{\mathbb{T}}) = 0$ and of *horizontal type* otherwise. In other words, ∂ is of fiber type if and only if the general orbits of corresponding $\mathbb{G}_a$-action on X are contained in the closures of $\mathbb{T}$-orbits. Recall that for a $\mathbb{T}$-action on an algebraic variety, the *complexity* is defined as the codimension of a general orbit. Toric varieties are $\mathbb{T}$-varieties of complexity zero. In this case a description of roots was obtained by M. Demazure [4], see also [3, Section 4]. There is a complete description of the homogeneous LNDs and roots in the case of $\mathbb{T}$-actions of complexity at most one due to A. Liendo, see [7]. A classification of homogeneous LNDs of fiber type in arbitrary complexity is given in [8]. The results of [7] and [8] are based on the combinatorial description of affine $\mathbb{T}$-varieties obtained by K. Altmann and J. Hausen, see [1].

Let $T \subset \mathbb{T}$ be a subtorus. The torus T also acts on X. Denote by M_T and $M_{\mathbb{T}}$ the character lattices of T and $\mathbb{T}$ respectively. It is obvious that if an LND respects the $M_{\mathbb{T}}$-grading on $\mathbb{K}[X]$, then it respects the M_T-grading as well. So any $\mathbb{T}$-root of a variety X can be restricted to some T-root. In this paper we show that any T-root of X arises in this way (Theorem 1). Also we study the restriction of roots in the case of affine toric varieties. For affine toric surfaces a complete description of this restriction is obtained.

In Sections 1 we recall some basic facts on locally nilpotent derivations and roots. Section 2 contains an elementary proof of surjectivity of restriction of roots. In Section 3 it is shown how the description of roots of the affine Cremona group given by A. Liendo in [9] can be obtained by our method. Note that Liendo's proof is based on the classification of LNDs of horizontal type in the case of $\mathbb{T}$-actions of complexity one. We give more direct proof. Section 4 is devoted to

the study of the restriction of roots on an affine toric variety. In Section 5 the case of affine toric surfaces is considered in more detail.

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero.

1. Homogeneous locally nilpotent derivations

Let X be an affine variety with an effective action of an algebraic torus $\mathbb{T}$ and $A = \mathbb{K}[X]$ be the algebra of regular functions on X. Denote by M the character lattice of the torus $\mathbb{T}$ and by N the lattice of one-parameter subgroups of $\mathbb{T}$. It is well known that there is a bijective correspondence between effective $\mathbb{T}$-actions on X and effective M-gradings on A. In fact, the algebra A is graded by a semigroup of lattice points in some convex polyhedral cone $\omega \subseteq M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ called the *weight cone*. So we have

$$A = \bigoplus_{m\in\omega_M} A_m \chi^m,$$

where $\omega_M = \omega \cap M$ and χ^m is the character corresponding to m.

A derivation ∂ on an M-graded algebra A is called *homogeneous* if it sends homogeneous elements to homogeneous ones. If $a, b \in A \setminus \ker\partial$ are some homogeneous elements, then $\partial(ab) = a\partial(b) + \partial(a)b$ is homogeneous too and hence $\deg\partial(a) - \deg a = \deg\partial(b) - \deg b$. So any homogeneous derivation ∂ has a well defined *degree* given as $\deg\partial = \deg\partial(a) - \deg a$ for any homogeneous $a \in A \setminus \ker\partial$. Homogeneous LNDs are called the *root vectors* with respect to the torus $\mathbb{T} = \operatorname{Spec}\mathbb{K}[M]$ and their degrees are said to be the $\mathbb{T}$-*roots*.

The following well known lemma shows that any LND decomposes into a sum of homogeneous derivations, some of which are locally nilpotent, cf. [7, Lemma 1.10].

Lemma 1. *Let A be a finitely generated M-graded domain. For any derivation ∂ on A there is a decomposition*

$$\partial = \sum_{e\in M} \partial_e, \tag{1}$$

where ∂_e is a homogeneous derivation of degree e. Moreover, the convex hull $\Delta(\partial) \subset M_{\mathbb{Q}}$ of the set $\{e \in M \mid \partial_e \neq 0\}$ is a polytope and for every vertex e of $\Delta(\partial)$, the derivation ∂_e is locally nilpotent if ∂ is.

Proof. For every homogeneous $a \in A$ there is a decomposition $\partial(a) = \sum_{e\in M} a_{\deg(a)+e}$, where the element $a_{\deg(a)+e}$ is homogeneous of degree $\deg(a) + e$. The linear map ∂_e given by the rule $\partial_e(a) = a_{\deg(a)+e}$ is a homogeneous derivation of degree e, and $\partial = \sum_{e\in M} \partial_e$. Since A is finitely generated, the set $\{e \in M \mid \partial_e \neq 0\}$ is finite and its convex hull $\Delta(\partial)$ is a polytope. Let $\widehat{e}$ be a vertex of $\Delta(\partial)$ and $a \in A_m$ be some homogeneous element. Then $\partial^n(a)$ has $\partial^n_{\widehat{e}}(a)$ as its homogeneous component of degree $m + n\widehat{e}$. So if ∂ is LND, then $\partial_{\widehat{e}}$ is LND as well. $\square$

Let X be an affine toric variety, i. e. a normal affine variety with a generically transitive action of a torus $\mathbb{T}$. In this case

$$A = \bigoplus_{m\in\omega_M} \mathbb{K}\chi^m = \mathbb{K}[\omega_M]$$

is the semigroup algebra. Recall that for given cone $\omega \subset M_{\mathbb{Q}}$, its *dual cone* is defined by

$$\sigma = \{n \in N_{\mathbb{Q}} \mid \langle n, p\rangle \geqslant 0 \ \ \forall p \in \omega\},$$

where $\langle , \rangle$ is the pairing between dual lattices N and M. Let $\sigma(1)$ be the set of rays of a cone σ and n_ρ be the primitive lattice vector on the ray ρ. By definition, for $\rho \in \sigma(1)$ set

$$S_\rho := \{e \in M \mid \langle n_\rho, e\rangle = -1 \text{ and } \langle n_{\rho'}, e\rangle \geqslant 0 \ \ \forall\, \rho' \in \sigma(1),\ \rho' \neq \rho\}.$$

One easely checks that the set S_ρ is infinite for any $\rho \in \sigma(1)$. The elements of the set $\mathfrak{R} := \bigsqcup_\rho S_\rho$ are called the *Demazure roots* of σ. This notion was introduced in [4]. Let $e \in S_\rho$. One can define

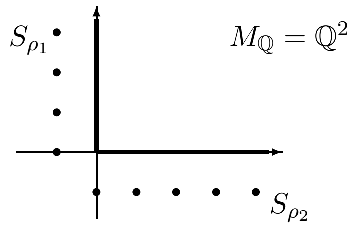
the homogeneous LND on the algebra A by the rule

$$\partial_e(\chi^m) = \langle n_\rho, m\rangle \chi^{m+e}.$$

It turns out that the set $\{\alpha\partial_e \mid \alpha \in \mathbb{K}, e \in \mathfrak{R}\}$ coincides with the set of all homogeneous LNDs on A, see [7, Theorem 2.7].

Example 1. Consider $X = \mathbb{A}^d$ with the standard action of the torus $(\mathbb{K}^\times)^d$. It is a toric variety with the cone $\sigma = \mathbb{Q}^d_{\geqslant 0}$ having rays $\rho_1 = \langle(1,0,\dots,0)\rangle_{\mathbb{Q}_{\geqslant 0}}, \dots, \rho_d = \langle(0,0,\dots,0,1)\rangle_{\mathbb{Q}_{\geqslant 0}}$. The dual cone ω is $\mathbb{Q}^d_{\geqslant 0}$ as well. In this case

$$S_{\rho_i} = \{(e_1,\dots,e_{i-1},-1,e_{i+1},\dots,e_d) \mid e_j \in \mathbb{Z}_{\geqslant 0}\}.$$



Denote $x_1 = \chi^{(1,0,\dots,0)}, \dots, x_d = \chi^{(0,\dots,0,1)}$. Then $\mathbb{K}[X] = \mathbb{K}[x_1,\dots,x_d]$. It is easy to see that the homogeneous LND corresponding to the root $e = (e_1,\dots,e_d) \in S_{\rho_i}$ is

$$\partial_e = x_1^{e_1}\dots x_{i-1}^{e_{i-1}} x_{i+1}^{e_{i+1}} \dots x_d^{e_d} \frac{\partial}{\partial x_i}.$$

2. Restriction of roots

Consider an affine $\mathbb{T}$-variety X and a subtorus $T \subset \mathbb{T}$. The torus T also acts on X. Evidently, every $\mathbb{T}$-homogeneous LND on $\mathbb{K}[X]$ is T-homogeneous and so any $\mathbb{T}$-root of X can be restricted to some T-root. The following theorem proves that every T-root arises in this way.

Theorem 1. *Let X be an affine variety with a regular action of a torus $\mathbb{T}$ and $T \subset \mathbb{T}$ be a subtorus. Then the restriction of $\mathbb{T}$-roots to T-roots is surjective. Moreover, if a T-root e is the restriction of the only one $\mathbb{T}$-root, then any T-homogeneous LND on $\mathbb{K}[X]$ of degree e is $\mathbb{T}$-homogeneous as well.*

Proof. Let e be a T-root of X and ∂ be a homogeneous LND of degree e. By Lemma 1, ∂ decomposes into a sum of $\mathbb{T}$-homogeneous derivations, some of which are locally nilpotent. So degree of any locally nilpotent summand is the $\mathbb{T}$-root of X and its restriction to T equals e. If e is the restriction of the only one $\mathbb{T}$-root, then there is only one summand in the decomposition of ∂, and hence ∂ is $\mathbb{T}$-homogeneous. □

Now we consider an affine toric variety X with the acting torus $\mathbb{T}$ and a subtorus $T \subset \mathbb{T}$ of codimension one. Let M be the character lattice of $\mathbb{T}$ and N be the lattice of one-parameter subgroups of $\mathbb{T}$. Denote by $\sigma_X \subset N_{\mathbb{Q}}$ the cone corresponding to X and by $\Gamma_T \subset N_{\mathbb{Q}}$ the hyperplane generated by one-parameter subgroups of $\mathbb{T}$ that are contained in the subtorus T.

Proposition 1. *Let X be an affine toric variety with an acting torus $\mathbb{T}$ and $T \subset \mathbb{T}$ be a subtorus of codimension one. If $\Gamma_T \cap \sigma_X = \{0\}$, then the restriction of roots is bijective. In particular, there is only one (up to scalar) root vector for each T-root e and any T-homogeneous LND on $\mathbb{K}[X]$ is $\mathbb{T}$-homogeneous as well.*

Proof. Let $\langle\cdot, m_T\rangle = 0$ be the equation of the hyperplane Γ_T, where $m_T \in M$. The vector m_T generates the kernel of the restriction of characters from $\mathbb{T}$ to T. Since $\Gamma_T \cap \sigma_X = \{0\}$, we may assume that $\langle p, m_T\rangle > 0$ for all $p \in \sigma_X \backslash \{0\}$. Suppose the restrictions of the roots e_1 and e_2 from $\mathbb{T}$ to T coincide and $e_1 - e_2 = \lambda m_T$, where $\lambda > 0$. Let ρ be the ray of σ_X with $e_1 \in S_\rho$. Then $\langle n_\rho, e_1 - e_2\rangle = -1 - \langle n_\rho, e_2\rangle \leqslant 0$. On the other hand, $\langle n_\rho, e_1 - e_2\rangle = \lambda\langle n_\rho, m_T\rangle > 0$. This

contradiction implies that the restriction of roots from $\mathbb{T}$ to T is injective. By Theorem 1, any T-homogeneous LND on $\mathbb{K}[X]$ is $\mathbb{T}$-homogeneous. □

Remark 1. In Proposition 1 we can not replace a subtorus of codimension one with a subtorus of arbitrary codimension. For example, let $X = \mathbb{A}^3$ and $T = \{(t, t, t^{-1}) \mid t \in \mathbb{T}^\times\}$. Denote by Γ_T the line corresponding to T. Then $\Gamma_T = \mathbb{Q} \cdot (1, 1, -1)$ and $\Gamma_T \cap \sigma_X = \{0\}$, but the restriction of roots is not bijective. Indeed, the T-homogeneous LND $\partial = x_2 x_3 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ is not $\mathbb{T}$-homogeneous.

3. Roots of the affine Cremona group

Consider the polynomial algebra $\mathbb{K}^{[n]} = \mathbb{K}[x_1, \dots, x_n]$. The set of "volume preserving" transformations

$$\operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]} = \{\gamma \in \operatorname{Aut}_{\mathbb{K}} \mathbb{K}^{[n]} \mid \det \left(\frac{\partial \gamma(x_j)}{\partial x_i} \right)_{1 \leqslant i,j \leqslant n} = 1\}$$

is a closed normal subgroup of the *affine Cremona group* $\operatorname{Aut}_{\mathbb{K}} \mathbb{K}^{[n]}$. It is an infinite dimensional simple algebraic group, see [11]. Every maximal algebraic torus in $\operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]}$ has dimension $n-1$ and is conjugate to

$$T = \{\gamma \in \operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]} \mid \gamma(x_i) = t_i x_i,\, t_i \in \mathbb{K},\, \prod_{i=1}^{n} t_i = 1\}.$$

If ∂ is an LND on $\mathbb{K}^{[n]}$, then $\exp t\partial \in \operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]}$ for any $t \in \mathbb{K}$, and so $\partial \in \operatorname{Lie}(\operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]})$. A nonzero LND ∂ of $\mathbb{K}^{[n]}$ is called a *root vector* of $\operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]}$ with respect to T if there exists a nontrivial character $\chi : T \to \mathbb{K}^\times$ such that

$$\gamma \circ \partial \circ \gamma^{-1} = \chi(\gamma)\partial \quad \forall\, \gamma \in T.$$

The character χ is said to be the *root* of $\operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]}$ with respect to T corresponding to ∂. It is easy to see that two last definitions are equivalent to the definitions of root vectors and roots given in Section 2 in the case $X = \mathbb{A}^n$. The first Popov's question in [10] was to find all roots and root vectors of $\operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]}$ with respect to T. The answer is given in [9]. It claims that the root vectors are exactly the LNDs $x^\alpha \frac{\partial}{\partial x_i}$, where x^α is any monomial not depending on x_i. Let us show that this statement immediately follows from Proposition 1. We have an affine toric variety $\mathbb{A}^n$ with the standard action of the torus $\mathbb{T} = (\mathbb{K}^\times)^n$ and the subtorus $T \subset \mathbb{T}$ given by equation $\prod_{i=1}^{n} t_i = 1$. The cone corresponding to $\mathbb{A}^n$ is just $\sigma_X = \mathbb{Q}^n_{\geqslant 0}$. The hyperplane Γ_T is given by equation $z_1 + \ldots + z_n = 0$ and we get $\Gamma_T \cap \sigma_X = \{0\}$. Thus all root vectors are homogeneous with respect to $\mathbb{T}$ and hence, as it is shown in Example 1, have the form $\lambda\, x^\alpha \frac{\partial}{\partial x_i}$, where $\lambda \in \mathbb{K}^\times$. So we obtain the following theorem.

Theorem 2. *The set of root vectors of* $\operatorname{Aut}^*_{\mathbb{K}} \mathbb{K}^{[n]}$ *with respect to* T *coincides with the set*

$$\{\lambda\, x^\alpha \frac{\partial}{\partial x_i} \mid \lambda \in \mathbb{K}^\times, i \in \{1, \dots, n\}, \alpha \in \mathbb{Z}^n_{\geqslant 0}, \alpha_i = 0\}.$$

The root corresponding to $\lambda\, x^\alpha \frac{\partial}{\partial x_i}$ *is the character* $\chi_{i,\alpha} : T \to \mathbb{K}^\times$ *given by* $\chi_{i,\alpha}(\gamma) = t_i^{-1} \prod_{j=1}^{n} t_j^{\alpha_j}$.

4. Subtorus actions on affine toric varieties

Let us start with three questions concerning roots on affine T-varieties. The first one was communicated to us by A. Liendo.

Question 1. *Let* e *be a root of an affine* T*-variety. How many root vectors correspond to the root* e*?*

As we have seen, in the toric case there is only one (up to scalar) root vector for each root. It follows from [8, Theorem 2.4] that root vectors of fiber type having degree e form the vector space (of finite or infinite dimension). For root vectors of horizontal type there is no complete answer even for T-action of complexity one.

Question 2. *Let X be an affine $\mathbb{T}$-variety and $T \subset \mathbb{T}$ be a subtorus. Consider a T-root e of X. How many $\mathbb{T}$-roots have e as their restrictions to T?*

Question 3. *Are all T-homogeneous LNDs of degree e on $\mathbb{K}[X]$ homogeneous with respect to $\mathbb{T}$?*

Theorem 1 shows that if a T-root e is the restriction of only one $\mathbb{T}$-root $\widehat{e}$, then any T-homogeneous LNDs of degree e is $\mathbb{T}$-homogeneous and there are as many root vectors corresponding to e as root vectors corresponding to $\widehat{e}$.

Let us investigate these questions in the toric case. Consider an affine toric variety X with the acting torus $\mathbb{T}$ and a subtorus $T \subset \mathbb{T}$ of codimension one. As above, denote by N the lattice of one-parameter subgroups, by M the character lattice of $\mathbb{T}$, by $\sigma_X \subset N_{\mathbb{Q}}$ the cone corresponding to X, and by Γ_T the hyperplane corresponding to subtorus T. Let $\langle \cdot, m_T \rangle = 0$ be the equation of the hyperplane Γ_T, where $m_T \in M$. The set of $\mathbb{T}$-roots of the variety X is the set $\mathfrak{R}_{\mathbb{T}} = \coprod_{\rho \in \sigma_X(1)} S_\rho$ of Demazure roots of the cone σ_X. Let $\mathfrak{R}_T$ be the set of T-roots and $\pi : \mathfrak{R}_{\mathbb{T}} \to \mathfrak{R}_T$ be the restriction of roots.

The answers to Questions 1-3 depend on relative position of the cone σ_X and the hyperplane Γ_T. We already have studied the case $\sigma_X \cap \Gamma_T = \{0\}$. By Proposition 1 there is only one root vector for every T-root e, only one $\mathbb{T}$-root has e as its restriction, and any T-homogeneous LND is $\mathbb{T}$-homogeneous as well. The following three propositions describe the restriction of roots in other cases.

Proposition 2. *If Γ_T intersects the interior of the cone σ_X and does not contain the rays of σ_X, then any T-root is the restriction of at most two $\mathbb{T}$-roots.*

Proof. Suppose that the restrictions of $\mathbb{T}$-roots $\widehat{e}_1 \in S_{\rho_1}, \widehat{e}_2 \in S_{\rho_2}$ and $\widehat{e}_3 \in S_{\rho_3}$ coincide. If $\rho_i = \rho_j$, then $\langle n_{\rho_i}, e_i - e_j \rangle = 0$ and the ray ρ_i is contained in Γ_T. So $\rho_i \neq \rho_j$ whenever $i \neq j$. We may assume without loss of generality that $\widehat{e}_1 - \widehat{e}_2 = \lambda m_T$ and $\widehat{e}_2 - \widehat{e}_3 = \mu m_T$, where $\lambda > 0$ and $\mu > 0$. Then $\lambda \langle n_{\rho_2}, m_T \rangle = 1 + \langle n_{\rho_2}, \widehat{e}_1 \rangle > 0$ and $\mu \langle n_{\rho_2}, m_T \rangle = -1 - \langle n_{\rho_2}, \widehat{e}_3 \rangle < 0$. So we get a contradiction. $\square$

Proposition 3. *Suppose that $\Gamma_T \cap \sigma_X$ is a face of the cone σ_X of positive dimension. Then*

a) $\pi(S_{\rho_1}) \cap \pi(S_{\rho_2}) = \varnothing$ *whenever* $\rho_1 \neq \rho_2$.

b) If a ray ρ is not contained in $\Gamma_T \cap \sigma_X$, then $\pi|_{S_\rho} : S_\rho \to \pi(S_\rho)$ is bijective.

c) If $\rho \subseteq \Gamma_T \cap \sigma_X$, then for any $e \in \pi(S_\rho)$ there are infinitely many elements of S_ρ, having e as their restriction, and root vectors corresponding to e form an infinite dimensional vector space.

Proof. a), b) If $\Gamma_T \cap \sigma_X$ is a face of the cone σ_X, then $\langle p, m_T \rangle \geqslant 0$ for all $p \in \sigma_X$. Suppose $\widehat{e}_1 \in S_{\rho_1}$, $\widehat{e}_2 \in S_{\rho_2}$, and $\widehat{e}_1 - \widehat{e}_2 = \lambda m_T$, where $\lambda > 0$. Then $\lambda \langle n_{\rho_1}, m_T \rangle = -1 - \langle n_{\rho_1}, \widehat{e}_2 \rangle \leqslant 0$ and $\lambda \langle n_{\rho_2}, m_T \rangle = 1 + \langle n_{\rho_2}, \widehat{e}_1 \rangle \geqslant 0$. It is possible if and only if $\rho_1 = \rho_2 \subseteq \Gamma_T \cap \sigma_X$.

c) Let $\widehat{e} \in S_\rho$, where $\rho \subseteq \Gamma_T \cap \sigma_X$. The vector $\widehat{e} + \lambda m_T$ is a root if and only if it belongs to the lattice and $\langle n_{\rho'}, \widehat{e} \rangle + \lambda \langle n_{\rho'}, m_T \rangle \geqslant 0$ for each $\rho' \neq \rho$. So there are infinite many roots, whose restrictions are equal to restriction of $\widehat{e}$. It remains to note that if $\widehat{e}_1, \widehat{e}_2 \in S_\rho$, then corresponding LNDs commute and hence their sum is again LND. $\square$

Proposition 4. *Suppose Γ_T intersects the interior of the cone σ_X and contains some rays of σ_X. Then*

a) If $\rho \not\subseteq \Gamma_T \cap \sigma_X$, then any $e \in \pi(S_\rho)$ is the restriction of at most two $\mathbb{T}$-roots.

b) If $\rho \subset \Gamma_T \cap \sigma_X$, then any $e \in \pi(S_\rho)$ is the restriction of a finite number k_e of $\mathbb{T}$-roots, there exists e with $k_e \geqslant 2$, and root vectors corresponding to e form a k_e-dimensional vector space.

Proof. a) Suppose that $e = \pi(\widehat{e}_1) = \pi(\widehat{e}_2)$, where $\widehat{e}_1 \in S_\rho$, $\widehat{e}_2 \in S_{\rho'}$, $\rho \not\subset \Gamma_T \cap \sigma_X$, and $\rho' \subset \Gamma_T \cap \sigma_X$. Then $\widehat{e}_1 - \widehat{e}_2 = \lambda m_T$ and hence $\langle n_{\rho'}, \widehat{e}_1\rangle = -1$. It is a contradiction. If $e \in \pi(S_{\rho_1}) \cap \pi(S_{\rho_2}) \cap \pi(S_{\rho_3})$, where $\rho_i \not\subset \Gamma_T \cap \sigma_X$, we can use the same arguments as in Proposition 2.

b) Let us show that any $e = \pi(\widehat{e})$, where $\widehat{e} \in S_\rho$ and $\rho \subset \Gamma_T \cap \sigma_X$, is the restriction of a finite number of elements. As above, the vector $\widehat{e} + \lambda m_T$ is a root if and only if it belongs to the lattice N and $\langle n_{\rho'}, \widehat{e}\rangle + \lambda\langle n_{\rho'}, m_T\rangle \geqslant 0$ for any $\rho' \neq \rho$. In this case there are rays ρ' with both positive and negative values $\langle n_{\rho'}, m_T\rangle$. Hence only a finite number of λ satisfies these conditions. In conclusion, let $\widehat{e} \in S_\rho$ be a root such that $\langle n_{\rho'}, \widehat{e}\rangle \geqslant -\langle n_{\rho'}, m_T\rangle$ whenever $\langle n_{\rho'}, m_T\rangle < 0$. Then $\widehat{e} + m_T$ is $\mathbb{T}$-root and $k_e \geqslant 2$ for $e = \pi(\widehat{e})$. □

Example 2. Consider $X = \mathbb{A}^3$ with the standard action of the torus $\mathbb{T} = (\mathbb{K}^\times)^3$. Let $T = \{(s_1, s_1, s_2) \mid s_1, s_2 \in \mathbb{K}^\times\}$ be a subtorus of codimension one. In this case the hyperplane Γ_T is generated by vectors $(1,1,0)$ and $(0,0,1)$. It can be easily checked that

$$\mathfrak{R}_T = \{(a,b), (c,-1) \mid a \in \mathbb{Z}_{\geqslant -1},\, b, c \in \mathbb{Z}_{\geqslant 0}\},$$

each root (a,b) is the restriction of two $\mathbb{T}$-roots, and each root $(c,-1)$ is the restriction of $c+1$ $\mathbb{T}$-roots.

Thus we have a complete description of the restriction of roots in the case, when $\sigma_X \cap \Gamma_T$ is a face of the cone σ_X. If the hyperplane Γ_T intersect the interior of σ_X, the answers to Questions 1 and 3 are remain unknown for T-roots $e \in \pi(S_{\rho_1}) \cap \pi(S_{\rho_2})$, where $\rho_1, \rho_2 \not\subset \Gamma_T$. The following proposition gives some sufficient condition on Γ_T, under which there are infinitely many root vectors corresponding to e. In Section 5 we show that this condition is also necessary in the case of affine toric surfaces.

Proposition 5. *Suppose that $e_1 \in S_{\rho_1}$, $e_2 \in S_{\rho_2}$, $\rho_1 \neq \rho_2$, $\pi(e_1) = \pi(e_2)$, and $\langle n_{\rho_1}, m_T\rangle = -1$. Then the linear map $\partial : \mathbb{K}[X] \to \mathbb{K}[X]$ given by*

$$\partial\chi^m = \chi^{m+e_2}(\alpha\langle n_{\rho_1}, m\rangle\chi^{m_T} + \beta\langle n_{\rho_2}, m\rangle)(\alpha\chi^{m_T} - \beta\langle n_{\rho_2}, m_T\rangle)^{\langle n_{\rho_1}, e_2\rangle}, \text{ for all } m \in \omega_M, \tag{2}$$

is a T-homogeneous LND of degree $\pi(e_1)$ for every $\alpha, \beta \in \mathbb{K}$.

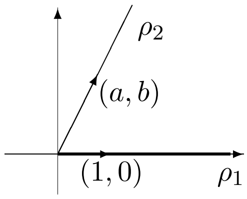
Proof. By assumptions, $e_1 - e_2 = (\langle n_{\rho_1}, e_2\rangle + 1)m_T$ and ∂ is a T-homogeneous derivation on A. By direct calculation we get

$$\partial^{k+1}\chi^m = \chi^{m+(k+1)e_2}(\alpha\chi^{m_T} - \beta\langle n_{\rho_2}, m_T\rangle)^{(k+1)\langle n_{\rho_1}, e_2\rangle} \sum_{j=0}^{k+1} \alpha^j \beta^{k-j+1} l_j^{(k+1)} \chi^{j m_T},$$

where $l_j^{(k+1)} = (\langle n_{\rho_1}, m\rangle - j + 1) l_{j-1}^{(k)} + (\langle n_{\rho_2}, m\rangle + j\langle n_{\rho_2}, m_T\rangle - k) l_j^{(k)}$, $l_1^{(1)} = \langle n_{\rho_1}, m\rangle$, and $l_0^{(1)} = \langle n_{\rho_2}, m\rangle$. Note that $l_j^{(k)} = 0$ whenever $k \geqslant \langle n_{\rho_2}, m\rangle + d\langle n_{\rho_2}, m_T\rangle + 1$ and $j \leqslant d$, and $l_i^{(k)} = 0$ whenever $k \geqslant \langle n_{\rho_1}, m\rangle + 1$ and $i \geqslant \langle n_{\rho_1}, m\rangle + 1$. Thus if $k = \langle n_{\rho_2}, m\rangle + \langle n_{\rho_2}, m_T\rangle\langle n_{\rho_1}, m\rangle$, then $\partial^{k+1}\chi^m = 0$, and ∂ is locally nilpotent. □

5. Affine toric surfaces

Normal affine surfaces with $\mathbb{C}^\times$-actions and LNDs on them were studied in [5], [6]. Here we consider a particular case. Namely, let X be an affine toric surface with the acting torus $\mathbb{T}$ and $T \subset \mathbb{T}$ be a one-dimensional subtorus. We give complete answers to Questions 1–3 for X. Let N_T and $N_{\mathbb{T}}$ be the lattices of one parameter subgroups of tori T and $\mathbb{T}$ respectively, and $\widehat{\sigma}_X \subset (N_{\mathbb{T}})_{\mathbb{Q}}$ be the cone corresponding to X. Up to automorphism of the lattice $N_{\mathbb{T}}$ we may assume that the cone $\widehat{\sigma}_X$ is generated by the vectors $n_{\rho_1} = (1,0)$ and $n_{\rho_2} = (a,b)$, where $a, b \in \mathbb{Z}_{\geqslant 0}$, $a < b$, and $\gcd(a,b) = 1$.



$\widehat{\sigma}_X \subset N_{\mathbb{Q}}$

The set of Demazure roots of $\widehat{\sigma}_X$ has the form:

$$S_{\rho_1} = \{(-1,m) \mid m \in \mathbb{Z},\, m \geqslant \frac{a}{b}\},\ \ S_{\rho_2} = \{(m_1,m_2) \mid m_1,m_2 \in \mathbb{Z},\, m_1 \geqslant 0,\, am_1 + bm_2 = -1\}.$$

Let Γ_T be the line corresponding to the subtorus T. There are three alternatives for relative position of the cone $\widehat{\sigma}_X$ and the line Γ_T:

(1) $\widehat{\sigma}_X \cap \Gamma_T = \{0\}$;
(2) $\widehat{\sigma}_X \cap \Gamma_T$ is a ray of $\widehat{\sigma}_X$;
(3) Γ_T intersects the interior of $\widehat{\sigma}_X$.

Case 1. Suppose $\widehat{\sigma}_X \cap \Gamma_T = \{0\}$. It follows from Proposition 1 that there is only one (up to scalar) root vector for each T-root e, only one $\mathbb{T}$-root has e as its restriction, and all T-homogeneous LNDs are $\mathbb{T}$-homogeneous as well. In the following figure we illustrate the T-roots of X. The restrictions of the elements of S_{ρ_1} and S_{ρ_2} are denoted by " $\bullet$ " and " $\circ$ " respectively.

Case 2. Assume that the intersection of the cone $\widehat{\sigma}_X$ with Γ_T is the ray ρ_1. In this case the restriction of the set S_{ρ_1} is just the point -1. By Proposition 3 the root vectors corresponding to $e = -1$ form an infinite dimensional vector space. For each root $e \neq -1$ there is only one $\mathbb{T}$- root, whose restriction equals e. The set of the T-roots looks like:

Case 3. Suppose Γ_T intersects the interior of the cone $\widehat{\sigma}_X$. Then the restriction π on each set S_{ρ_i} is injective and the number $D = rb - qa$ is positive for primitive lattice vector (r,q) of the line Γ_T. Let us prove the following lemma.

Lemma 2. *Suppose the line Γ_T intersects the interior of the cone $\widehat{\sigma}_X$. Then the intersection $\pi(S_{\rho_1}) \cap \pi(S_{\rho_2})$ is not empty if and only if $a-1$ is divisible by* $\gcd(q, rb-qa)$.

Proof. The restrictions of S_{ρ_1} and S_{ρ_2} have the form $\{-r + mq \mid m \in \mathbb{Z}_{\geqslant \frac{a}{b}}\}$ and $\{rm_1^0 + qm_2^0 + kD \mid k \in \mathbb{Z}_{\geqslant 0}\}$, where $(m_1^0, m_2^0) \in S_{\rho_2}$ is a root, respectively. The intersection of these two sets is non-empty if and only if there exist numbers m_0 and k_0 such that

$$r + rm_1^0 + qm_2^0 = m_0 q - k_0 D. \tag{3}$$

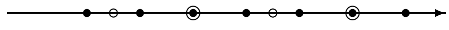
Multiplying this equation by a and using the fact that $(m_1^0, m_2^0) \in S_{\rho_2}$, we get

$$r(a-1) - Dm_2^0 = m_0 qa - k_0 Da. \tag{4}$$

So if $a-1$ is not divisible by $\gcd(q,D)$, we obtain a contradiction. Conversely, suppose that the left part of (4) is divisible by $\gcd(q,D)$. Since b is divisible by $\gcd(q,D)$ and $\gcd(a,b)=1$, the left part of (3) is also divisible by $\gcd(q,D)$. Hence there exist m_0 and k_0 such that (3) holds. $\square$

Let us draw the set of the T-roots as above.

a) $\gcd(q, rb-qa) \nmid a-1$

$b)$  $\gcd(q, rb - qa) \mid a-1$

In case $a)$ the restriction of roots from $\mathbb{T}$ to T is bijective and any T-homogeneous LND is $\mathbb{T}$-homogeneous. In case $b)$ denote by Λ the intersection $\pi(S_{\rho_1}) \cap \pi(S_{\rho_2})$. For the roots $e \notin \Lambda$ we can use Theorem 1. It remains to answer Questions 1 and 3 for $e \in \Lambda$.

Now we recall a description of T-varieties of complexity one. Let N and M be two mutually dual lattices with the pairing denoted by $\langle , \rangle$, σ be a cone in $N_{\mathbb{Q}}$, $\omega \subset M_{\mathbb{Q}}$ be its dual cone, and C be a smooth curve. Consider a divisor $\mathfrak{D} = \sum\limits_{z \in C} \Delta_z \cdot z$, whose coefficients Δ_z are polyhedra in $N_{\mathbb{Q}}$ with tail cone σ. Such objects are called *σ-polyhedral divisors*, see [1]. For every $m \in \omega_M$ we can get the $\mathbb{Q}$-divisor $\mathfrak{D}(m) = \sum\limits_{z \in C} \min\limits_{p \in \Delta_z} \langle p, m\rangle \cdot z$. One can define the M-graded algebra

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \omega_M} A_m \chi^m, \text{ where } A_m = H^0(C, \mathfrak{D}(m))$$

and the multiplication is determined in natural way. It follows from [1] that $A[C, \mathfrak{D}]$ is a normal affine domain and that every normal affine M-graded domain A with $\operatorname{tr.deg.} A = \operatorname{rk} M + 1$ is equivariantly isomorphic to $A[C, \mathfrak{D}]$ for some C and $\mathfrak{D}$.

Also we need a description of homogeneous LNDs of horizontal type for $\mathbb{T}$-variety X of complexity one. Below we follow the approach given in [2]. We have $\mathbb{K}[X] = A[C, \mathfrak{D}]$ for some C and $\mathfrak{D}$. It turns out that C is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ whenever there exists a homogeneous LND of horizontal type on $A[C, \mathfrak{D}]$, see [7, Lemma 3.15]. Let C be $\mathbb{A}^1$ or $\mathbb{P}^1$, $\mathfrak{D} = \sum\limits_{z \in C} \Delta_z \cdot z$ a σ-polyhedral divisor on C, $z_0 \in C$, $z_\infty \in C \backslash \{z_0\}$, and v_z a vertex of Δ_z for every $z \in C$. Put $C' = C$ if $C = \mathbb{A}^1$ and $C' = C \backslash \{z_\infty\}$ if $C = \mathbb{P}^1$. A collection $\widetilde{\mathfrak{D}} = \{\mathfrak{D}, z_0; v_z, \forall z \in C\}$ if $C = \mathbb{A}^1$ and $\widetilde{\mathfrak{C}} = \{\mathfrak{C}, z_0, z_\infty; v_z, \forall z \in C'\}$ if $C = \mathbb{P}^1$ is called a *colored* σ-polyhedral divisor on C if the following conditions hold:

(1) $v_{\deg} := \sum\limits_{z \in C'} v_z$ is a vertex of $\deg \mathfrak{D}|_{C'} := \sum\limits_{z \in C'} \Delta_z$; (2) $v_z \in N$ for $z \neq z_0$.

Let $\widetilde{\mathfrak{D}}$ be a colored σ-polyhedral divisor on C and $\delta \subseteq N_{\mathbb{Q}}$ be the cone generated by $\deg \mathfrak{D}|_{C'} - v_{\deg}$. Denote by $\widetilde{\delta} \subseteq (N \oplus \mathbb{Z})_{\mathbb{Q}}$ the cone generated by $(\delta, 0)$ and $(v_{z_0}, 1)$ if $C = \mathbb{A}^1$, and by $(\delta, 0)$, $(v_{z_0}, 1)$ and $(\Delta_{z_\infty} + v_{\deg} - v_{z_0} + \delta, -1)$ if $C = \mathbb{P}^1$. By definition, put d the minimal positive integer such that $d \cdot v_{z_0} \in N$. A pair $(\widetilde{\mathfrak{D}}, e)$, where $e \in M$, is said to be *coherent* if

(1) There exists $s \in \mathbb{Z}$ such that $\widetilde{e} = (e, s) \in M \oplus \mathbb{Z}$ is a Demazure root of the cone $\widetilde{\delta}$ with distinguished ray $\widetilde{\rho} = (d \cdot v_{z_0}, d)$.

(2) $v(e) \geqslant 1 + v_z(e)$ for every $z \in C' \backslash \{z_0\}$ and every vertex $v \neq v_z$ of the polyhedron Δ_z.

(3) $d \cdot v(e) \geqslant 1 + v_{z_0}(e)$ for every vertex $v \neq v_{z_0}$ of the polyhedron Δ_{z_0}.

(4) If $Y = \mathbb{P}^1$, then $d \cdot v(e) \geqslant -1 - d \cdot \sum\limits_{z \in Y'} v_z(e)$ for every vertex v of the polyhedron Δ_{z_∞}.

It follows from [2, Theorem 1.10] that homogeneous LNDs of horizontal type on $A[C, \mathfrak{D}]$ are in bijection with the coherent pairs $(\widetilde{\mathfrak{D}}, e)$.

Let us return to our case. Following [1, Section 11] we will show how to determine C and $\mathfrak{D}$ such that $X \overset{T}{\cong} \operatorname{Spec} A[C, \mathfrak{D}]$. Since $T \subset \mathbb{T}$, we have an exact sequence

$$0 \longrightarrow N_T \xrightarrow{F} N_{\mathbb{T}} \xrightarrow{P} N_{\mathbb{T}}/N_T \longrightarrow 0.$$

Let Σ be the coarsest quasifan in $(N_{\mathbb{T}}/N_T)_{\mathbb{Q}}$ refining all cones $P(\tau)$, where τ runs over all faces of $\widehat{\sigma}_X$. It follows from [1, Section 6] that the desired curve C is the toric variety corresponding to Σ. Let us choose the projection $s : N_{\mathbb{T}} \to N_T$, which satisfies $s \circ F = id$. The divisor $\mathfrak{D}$ on C such that $X \overset{T}{\cong} \operatorname{Spec} A[Y, \mathfrak{D}]$ is $\mathfrak{D} = \sum_{\rho \in \Sigma(1)} \Delta_\rho \cdot D_\rho$, where $\Sigma(1) \subset \Sigma$ is the set of one-dimensional cones, D_ρ is the prime divisor corresponding to ρ, $\Delta_\rho = s(\widehat{\sigma}_X \cap P^{-1}(n_\rho)) \subset (N_T)_{\mathbb{Q}}$, and n_ρ is the

primitive lattice vector on ρ. Here all Δ_ρ are σ-tailed polyhedra with $\sigma = s(\widehat{\sigma}_X \cap (F(N_T))_{\mathbb{Q}}$. In our case

$$F = \begin{pmatrix} r \\ q \end{pmatrix}, \ s = (u, v) \ \text{and} \ P = (q, -r),$$

where $u, v \in \mathbb{Z}$, $ru + qv = 1$. By direct calculation we obtain $Y = \mathbb{P}^1$, $\sigma = \mathbb{Q}_{\geqslant 0}$, and

$$\mathfrak{D} = (p_1 + \sigma) \cdot [0] + (p_2 + \sigma) \cdot [\infty],$$

where $p_1 = \dfrac{u}{q}$ and $p_2 = \dfrac{au + bv}{rb - qa}$. In this case there is no homogeneous LND of fiber type on $A[C, \mathfrak{D}]$. Homogeneous LNDs of horizontal type of degree e are in bijection with the coherent pairs $(\widetilde{\mathfrak{D}}, e)$. If neither p_1 nor p_2 belong to $\mathbb{Z}$, then we should put either $z_0 = 0$ and $z_\infty = \infty$, or $z_0 = \infty$ and $z_\infty = 0$ in the definition of colored σ-polyhedral divisor $\widetilde{\mathfrak{D}}$. It means that there exist at most two LNDs of degree e. So for a root $e \in \Lambda$ there are two root vectors and they are $\mathbb{T}$-homogeneous. It is clear that p_1 is integer if and only if q equals 1. Let us find out when p_2 is integer.

Lemma 3. *The number* $p_2 = \dfrac{au + bv}{rb - qa}$ *is integer if and only if* $rb - qa = 1$.

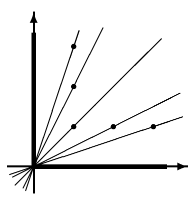
Proof. Denote $D = rb - qa$. Suppose $au + bv = kD$, where $k \in \mathbb{Z}$. We also know that $ru + qv = 1$. Using these equations, we get

$$a = aru + aqv = D(rk - v), \quad b = bru + bqv = D(qk - u).$$

So a and b are divisible by D. This holds if and only if $D = 1$. □

Now suppose $p_1 \notin \mathbb{Z}$ and $p_2 \in \mathbb{Z}$. Then for any colored divisor either $z_0 = 0$ or $z_\infty = 0$ holds. It is not hard to check that in the former case the pair $(\widetilde{\mathfrak{D}}, e)$ is coherent if and only if $\frac{1}{q} + ep_1 \in \mathbb{Z}$ and $qep_2 + 1 \geqslant 0$. These conditions do not depend on the choice of z_∞. So if there is one coherent pair $(\widetilde{\mathfrak{D}}, e)$ with $z_0 = 0$, then any pair $(\widetilde{\mathfrak{D}}', e)$ with $z_0 = 0$ and $z_\infty \in \mathbb{P}^1 \setminus \{z_0\}$ is coherent. Since $p_2 \in \mathbb{Z}$, we consider all $\widetilde{\mathfrak{D}}$ with $z_\infty = 0$ as the same colored σ-polyhedral divisor. Let $e \in \Lambda$. We already know two root vectors of degree e, which are $\mathbb{T}$-homogeneous LNDs, whose degrees belong to different S_{ρ_i}. Hence the inequality $z_0 \neq z_0'$ holds for corresponding coherent pairs $(\widetilde{\mathfrak{D}}, e)$ and $(\widetilde{\mathfrak{D}}', e)$. Thus there are as many root vectors corresponding to e as points on $\mathbb{P}^1$. All these root vectors are described by Proposition 5. The cases $p_1 \in \mathbb{Z}$, $p_2 \notin \mathbb{Z}$ and $p_1, p_2 \in \mathbb{Z}$ are analyzed similarly.

The following figure illustrates the lines Γ_T with $p_1 \in \mathbb{Z}$ or $p_2 \in \mathbb{Z}$.



Summarizing, we obtain the following statement.

Proposition 6. *Let X be an affine toric surface with the acting torus $\mathbb{T}$ and $T \subset \mathbb{T}$ be a one-dimensional subtorus. Suppose that the cone $\widehat{\sigma}_X$ corresponding to X is generated by $n_{\rho_1} = (1, 0)$ and $n_{\rho_2} = (a, b)$, where $a, b \in \mathbb{Z}_{\geqslant 0}$, $a < b$, and $\gcd(a, b) = 1$, and the line Γ_T is generated by (r, q), where $\gcd(r, q) = 1$. Let π be the restriction of roots from $\mathbb{T}$ to T, e be a T-root, $\Lambda = \pi(S_{\rho_1}) \cap \pi(S_{\rho_2})$, and $D = rb - qa$. Then the following table describes the restriction of roots.*

	The number of root vectors corresponding to e	*The number of $\mathbb{T}$-roots having e as their restriction*	*Are all T-homogeneous LNDs on X $\mathbb{T}$-homogeneous?*
1. $\widehat{\sigma}_X \cap \Gamma_T = \{0\}$	*1*	*1*	*yes*
2. $\widehat{\sigma}_X \cap \Gamma_T = \mathbb{Q}_{\geqslant 0} \cdot \rho_i$			
$e = -1$	*an infinite dimensional space*	∞	*no*
$e \neq -1$	*1*	*1*	*yes*
3. $\widehat{\sigma}^{\circ}_X \cap \Gamma_T \neq \varnothing$			
3.1. $\gcd(q, D) \nmid a - 1$	*1*	*1*	*yes*
3.2. $q \neq 1, D \neq 1$ *and* $\gcd(q, D) \mid a - 1$			
$e \notin \Lambda$	*1*	1	*yes*
$e \in \Lambda$	*2*	*2*	*yes*
3.3. $q = 1$ *or* $D = 1$			
$e \notin \Lambda$	*1*	1	*yes*
$e \in \Lambda$	*as many as points on* $\mathbb{P}^1$	*2*	*no*

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